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Formulation of the Ablation Thermal Problem in a Unified Form

F. A. A. Gomes,* J. B. Campos Silva,† and A. J. Diniz‡
State University of São Paulo,
15 385-000 São Paulo, Brazil

Nomenclature

K	=	radius of curvature of a surface
$Q(\tau)$	=	dimensionless heat flux, $k(T_f - T_0)/L$
R	=	circle radius
T_f	=	melting temperature
t	=	dimensional time
t_c	=	characteristic time, L^2/α
t_f	=	time of starting melting
t_r	=	reference time
α	=	thermal diffusivity
β	=	is a constant value
Γ	=	auxiliary modified coordinate (material not consumed by the ablation process) for the case of the flat plate
γ	=	angle among the tangent, the surface and normal to the radius of curvature of the curvilinear surface
δ	=	thickness of the material
ζ	=	dimensionless coordinate on the normal of the surface normalized by the thickness of the wall of the body, η/δ
η	=	normal vector to the surface
θ	=	is the dimensionless temperature, $(T - T_0)/(T_f - T_0)$
ν	=	inverse of the Stefan number, $H/c(T_f - T_0)$
ξ	=	variable that characterizes the orientation of the system of the curvilinear geometry
τ	=	dimensionless time, t/t_c
τ_f	=	dimensionless time of staring melting, t_m/t_c
τ_r	=	dimensionless reference time, t_r/t_c

I. Introduction

ENGINEERING applications involving high-speed atmospheric flight or high-temperature propulsive devices such as reentry bodies or rocket nozzles created a need for high-temperature thermal protection systems. Ablative systems have been found to offer a practical, often indispensable, solution to many reentry applications problems.

In contrast to other subjects, it is not possible to make a definitive recommendation of formulas, equations, and definitive solutions to the various transfer regimes encountered in ablation. This is due to the complexity of the ablation process, which involves heat and mass transfer interactions between the environmental medium and a solid but decomposing object over which it flows, with the attendant chemical reactions and changes in phase.

The analytical and numerical solutions, as well as analytical–numerical solutions, have been obtained. In a one-dimensional geometry, the method of approximate integral balance was presented by Chung and Hsiao.[1] The solution of the diffusion problem with variable coefficients was studied by Özisik and Cotta.[2] A generalized study of the ablative phenomenon was made by Adams[3] and Sutton.[4] The physical and mathematical models of the ablation process were presented by Lacaze[5] and Zaparoli (E. L. Zaparoli, private communication, Oct. 1989). Cotta,[6] Diniz et al.,[7–10] and Gomes et al.[11] presented the use of the generalized integral transform technique (GITT). They solved the one-dimensional problems of heat diffusion for several geometries. Vallerani[12] applied the integral method to problems of simple classes of ablation. The classical integral transform Technique for linear problems was presented by Mikhailov and Özisik.[13] The GITT is a generalization of that technique to nonlinear problems, as presented by Cotta.[14]

In the present work a one-dimensional analysis is accomplished in a curvilinear geometry of revolution using a hybrid numerical–analytical model approach to solve a problem with variable domain, similar to the ablative problem, by the application of GITT. To solve the problem a simplified mathematical model was used, in which a prescribed heat flux replaced the effects of the aerodynamic heating imposed by the external flow. This simplification is used as a preliminary analysis for future work involving systems with variable domain under more realistic conditions, as well as to show the application of a generalized equation to generate results for several geometries.

II. Mathematical Analysis

In this work, a problem of heat transfer in an orthogonal curvilinear coordinate system with an initially uniform temperature T_0 and with constant physical properties is considered. A continuous function $\xi(\zeta)$ that conveys the information of the geometry of the studied body is defined in a such way that it can be applied to solve the same problem in all of the geometries contained in Table 1.

The bodies are subjected to a time-dependent prescribed heat flux at one face and insulated on the other. After an initial heating period (preablation period) until the instant τ_f, the ablation starts at the heated surface through melting and continuous removal of the surface material (ablation period), yielding an inward motion of the boundary at a rate not known a priori.

Table 1 Values of the parameter ξ in agreement with the generalization of the coordinates

	Values for $\xi(\zeta)$			
Geometry	R	β	K	$\cos\gamma$
Flat plate	∞	0	0	0
Cylindrical	∞	1	0	0
Spherical	∞	2	0	0
Revolution	R	0	K	$\cos\gamma$

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*Graduate Student, Department of Mechanical Engineering, Faculty of Engineering, Av. Brazil Centro, 56, Ilha Solteira; francisco@dem.feis.unesp.br.

†Assistant Professor, Department of Mechanical Engineering, Av. Brazil Centro, 56, Ilha Solteira; jbcampos@dem.feis.unesp.br.

‡Assistant Professor, Department of Mechanical Engineering, Av. Brazil Centro, 56, Ilha Solteira; diniz@dem.feis.unesp.br.

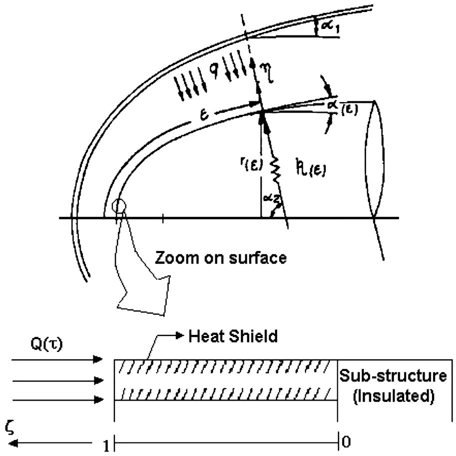

Fig. 1 Curvilinear coordinates in the dimensionless form.

Figure 1 shows the geometry and the physical situation of the phenomenon imposed in the domain.

A. Preablation Period

In dimensionless form, the diffusion equation that models the phenomenon in the preablation period in the variable used to homogenize the boundary condition is shown in Ref. 10. This variable is defined as $\theta^*(\zeta,\tau) = \theta(\zeta,\tau) + (\zeta^2/2)Q(\tau)$, and $\xi(\zeta)$ is given by

$$\xi(\zeta) = \left[\frac{\beta}{\zeta} + \frac{k}{1+k\zeta} + \frac{\cos\gamma}{R+\zeta\cos\gamma}\right] \tag{1}$$

Equation (1) it is the generalized equation for the results in the several geometries indicated in Table 1. In Table 1 are presented the parameters according to the desired geometry.

In the variable θ^* the initial and boundary conditions become

$$\theta^*(0,\tau) = \frac{\zeta^2}{2}Q(0) \tag{2}$$

$$\left.\frac{\partial\theta^*(\zeta,\tau)}{\partial\zeta}\right|_{\zeta=0} = 0, \qquad \left.\frac{\partial\theta^*(\zeta,\tau)}{\partial\zeta}\right|_{\zeta=1} = 0 \tag{3}$$

Applying all steps of the integral transform technique[10,13] for solution of the proposed problem, it is possible to obtain the integral transform

$$\tilde{\theta}_i^*(\tau) = \int_0^1 \psi(\mu_i,\zeta)\theta^*(\zeta,\tau)\,\mathrm{d}\zeta, \qquad \text{transform} \tag{4}$$

$$\theta^*(\zeta,\tau) = \sum_{i=0}^{\infty}\frac{\psi(\mu_i,\zeta)}{N_i}\tilde{\theta}_i^*(\tau), \qquad \text{inverse} \tag{5}$$

where N_i is the normalization integral, as presented in Refs. 13 and 14.

Following the steps by the application of the technique,[13] it is possible to obtain the equation for the temperature field $\theta(\zeta,\tau)$ of the preablation period in the form

$$\theta(\zeta,\tau) = \sum_{i=1}^{\infty}\frac{\psi_i(\zeta)}{N_i}\tilde{\theta}_i^*(\tau) + \frac{\zeta^2}{2}Q(\tau) + \theta_m(\tau) \tag{6}$$

where $\psi(\zeta) = \psi(\mu_i,\zeta) = \cos(\mu_i\zeta)$ are the eigenfunctions, $\mu = \mu_i = i\pi (i = 1, 2, \ldots)$ are the eigenvalues, and $Q(\tau) = a + b\tau + c\tau^2$ is the time-dependent polynomial prescribed heat flux and $\theta_m(\tau)$ is the mean temperature potential, given by

$$\theta_m(\tau) = \left(\frac{5}{6} + \int_0^1 \zeta\xi(\zeta)\,\mathrm{d}\zeta\right)\left(a\tau + \frac{b\tau^2}{2} + \frac{c\tau^3}{3}\right) - \frac{1}{6}(b\tau + c\tau^2) \tag{7}$$

where a, b, and c are known constants.

The time of the start of melting, τ_f, is obtained by the solution of the equation

$$\theta(\zeta,\tau_f) \equiv \sum_{i=1}^{\infty}4\cos(i\pi)\tilde{\theta}_i^*(\tau_f) + Q(\tau_f) = 2 - \theta_m(\tau_f) \tag{8}$$

for $\zeta = 1$, with the temperature in the ablative boundary, $\theta(1,\tau_f) = 1$, which means that the corresponding time to the ablation starting was reached.

The solution of the problem in the ablative period involves the determination of the position of the moving contour. This position is identified by the thickness $\delta(\tau)$ of the material that has not been melted yet. The melted part is represented by $s = 1 - \delta(\tau)$. The solution of the preablation period is the initial condition for the ablation period. The same equation as for the preablation period is used in the ablation period. To use the adopted technique, the change of variables $\hat{\theta}(\zeta,\tau) = \theta(\zeta,\tau) - 1$ is used in the equation of the preablation period.

B. Ablation Period

The equation that will govern the problem in this phase, $\tau > \tau_f$, is

$$\frac{\partial\hat{\theta}(\zeta,\tau)}{\partial\tau} = \frac{\partial^2\hat{\theta}(\zeta,\tau)}{\partial\zeta^2} + \xi(\zeta)\frac{\partial\hat{\theta}(\zeta,\tau)}{\partial\zeta}, \qquad \begin{matrix} 0 < \zeta < \delta(\tau) \\ \tau > \tau_f \end{matrix} \tag{9}$$

with initial and boundary conditions

$$\hat{\theta}(\zeta,\tau_f) = \theta(\zeta,\tau_f) - 1, \qquad \tau = \tau_f \tag{10}$$

$$\left.\frac{\partial\hat{\theta}(\zeta,\tau)}{\partial\zeta}\right|_{\zeta=0} = 0, \qquad \hat{\theta}(\zeta,\tau)\big|_{\zeta=\delta(r)} = 0 \tag{11}$$

To determine the thickness and the rate of loss of the ablative material, the following additional equation resulting from the energy balance at the contour is used:

$$\frac{\partial\hat{\theta}(\zeta,\tau)}{\partial(\zeta)} = \nu\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} + Q(\tau), \qquad \zeta = \delta(\tau) \tag{12}$$

Applying the technique in the same way as in the preablation period in Eq. (10) (Refs. 10 and 13) the transformed initial conditions are obtained by the temperature field of the ablation period, when $\tau = \tau_f$:

$$\tilde{\hat{\theta}}_i(\tau_f) = \frac{4\sqrt{2}}{\pi}(2i-1)(-1)^{i+1}\sum_{k=1}^{\infty}\tilde{\theta}_k^*(\tau_f)\frac{(-1)^k}{(2i-1)^2 - 4k^2} + \frac{2\sqrt{2}}{\pi}\frac{(-1)^{i+1}}{(2j-1)}\left\{[\theta_m(\tau_f) - 1] + \frac{Q(\tau_f)}{2}\left[1 - \frac{2^3}{(2i-1)^2}\right]\right\} \tag{13}$$

The complementary equation, Eq. (12), in the transformed form becomes, for $\tau > \tau_f$,

$$\frac{\mathrm{d}\delta(\tau)}{\mathrm{d}\tau} = \frac{\sqrt{2}}{2\nu}\pi\left(\frac{1}{\delta(\tau)}\right)^{\frac{3}{2}}\sum_{i=1}^{\infty}(2i-1)\tilde{\hat{\theta}}_i(\tau)(-1)^i - \frac{Q(\tau)}{\nu} \tag{14}$$

After application of GITT, Eqs. (15) and (16) form a infinite system of ordinary differential equations that have been truncated in some order N for solution:

$$\frac{\mathrm{d}\tilde{\hat{\theta}}_i(\tau)}{\mathrm{d}\tau} + \lambda_i^2(\tau)\,\tilde{\hat{\theta}}_i(\tau) + \sum_{j=1}^{N}A_{ij}^*(\tau)\tilde{\hat{\theta}}_j(\tau) = 0 \tag{15}$$

$$\frac{\mathrm{d}\delta(\tau)}{\mathrm{d}(\tau)} = \frac{\sqrt{2}}{2\nu}\pi\left(\frac{1}{\delta(\tau)}\right)^{\frac{3}{2}}\sum_{i=1}^{N}(2i-1)\tilde{\hat{\theta}}_i(\tau)(-1)^i - \frac{Q(\tau)}{\nu} \tag{16}$$

where the terms A_{ij} and the integral transform $(\tilde{\hat{\theta}})$ are shown in Refs. 10 and 13.

The initial condition for the solution of the system of Eqs. (15) and (16) is Eq. (11). The solution supplies the values of the thickness and of the rate of loss of the ablative material (ablation rate), as well as the temperature distribution.

III. Results

This section presents the results obtained in the present work and compares them with results available in the literature. The results of interest are the depth of ablation, the ablation rate, and the temperature field. In this work all the physical parameters considered were the same as used in Ref. 1: thermal diffusivity $\alpha = 0.00929$ m^2/s; characteristic time $t_c = 10$ s; characteristic length $L = 0.3048$ m; reference heat flux $q_r = 113{,}568.89$ W/m^2; temperature difference $T_f - T_0 = 37.78$°C; inverse of the Stefan number $\nu = 1$; and reference time $t_r = 100$ s.

Using the equation that represents the preablation period and varying the parameters $\xi(\zeta)$, the solution for any mentioned geometry in Table 1 can be obtained.

Figure 2 shows the temperature field for a constant heat flux, where it is possible to notice the temperature increase with the time. The start of the ablation period is at the time $\tau = \tau_f = 0.197$ and is in agreement with Ref. 7.

Figure 3 shows the depth $S(\tau)$ and the ablation speed $V(\tau)$, when $\xi(\zeta)$ is equal to zero, simulating the case of Ref. 7. The curves show the solution of the problem by five methods: finite differences (FDM), integral of the heat balance method (IHBM), θ-moment integral (θ-MIM), GITT (flat plate), and W-GITT (generalized equation). In Fig. 3 is also shown a comparison among the three mentioned cases,[1] where it is possible to observe that using Eq. (1) of the present work (generalized equation in the graph: W-GITT) it is possible to reproduce the same points obtained with the case of the flat plate[7] (flat plate in the graph: GITT). The GITT is able to reproduce close results for the case of the approximate solution presented in Ref. 1 and also in Ref. 7 when the generalized equation is used.

One can conclude that the generalized equation is able to produce good results with the application of GITT for simulating the ablation in a flat plate using the characteristic values of Table 1.

Figure 4 shows a simulation using the generalized equation for the cases of cylindrical and spherical geometries[8,9] when $\beta = 1$ and 2, respectively, for radius $R = 25$ and heat flux $Q(t) = 10t$.

Thus, it is possible to notice that the results obtained with the generalized equation agree with the results presented in the mentioned references. The generalized equation also reproduced results for the cylindrical and spherical geometries.

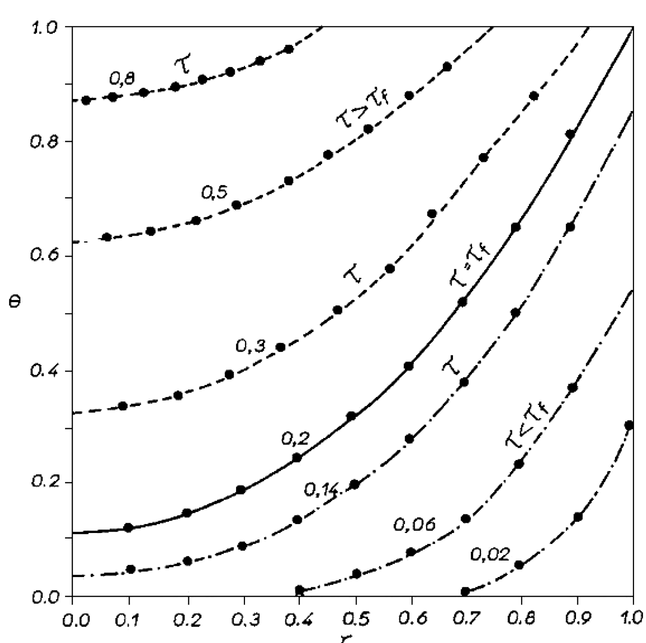

Fig. 2 Temperature distribution for $\xi(\zeta) = 0$ and heat flux $Q(\tau) = 2$.

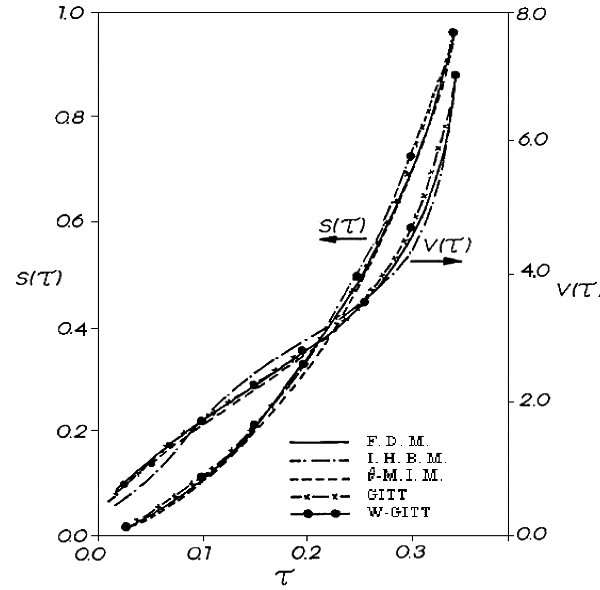

Fig. 3 Ablation depth and ablation rate for $\xi(\zeta) = 0$ and heat flux $Q(\tau) = 10\tau^2$.

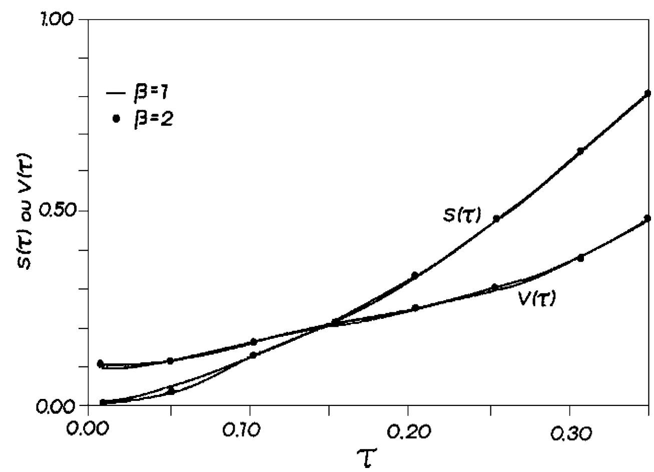

Fig. 4 Ablation depth and ablation rate for $\xi(\zeta) = \beta/\zeta$; $\beta = 1$, $\beta = 2$, $K = 0$, $\cos\gamma = 0$, and heat flux $Q(\tau) = 10\tau$.

In relation to Fig. 3 it is observed that the IBC method presents a sensible deviation of the results of the ablation speed, in relation of the results obtained by GITT. The method FDM seems to be accurate for many practical proposals, except for small intervals of time at the beginning of the ablation. GITT presented good results in the whole time interval. The θ-moment integral method is an approach that improves considerably the IBC method.

GITT is a hybrid analytical–numerical method,[13] where the solutions are based on coupled ordinary differential equation systems. With good control of the precision in the convergence, it implies more accurate, more exact results, when compared with methods A, B, and C. It is also possible to notice that with the different imposed heat flux, the results were not affected compared with the application of the generalized equation in mentioned references.

IV. Conclusions

The application of the GITT in the generalized equation for several geometries gives satisfactory accuracy when compared with the case of Ref. 1, and more accuracy results when compared with other solution methods for problems that simulate approximated cases of the ablative phenomenon in a flat plate geometry.

The generalized equation with the application of GITT was able to reproduce any of the geometries presented in Table 1, when compared with those of Refs. 7–9.

The objective of trying to propose an approximated solution to validate GITT in a simulation, as proposed in Ref. 1, confirms that GITT is able to reproduce as good or better results than those presented in the literary, as is the case of the methods presented in Ref. 1. And further, if GITT can be applied to solve a case with a generalized equation reproducing the same results from Ref. 1, we believe that GITT is able to solve problems under more realistic conditions involving the case of variable boundary present in the ablative phenomenon. This more realistic situation, in which the velocity field also has to be solved, is already in process in another work.

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T. Lin
Associate Editor